

STRINGY SPHALERONS AND NON-ABELIAN BLACK HOLES

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Abstract

Static spherically symmetric asymptotically flat particle-like and black hole solutions are constructed within the $SU(2)$ sector of the 4-dimensional heterotic string effective action. They separate topologically distinct Yang-Mills vacua and are qualitatively similar to the Einstein-Yang-Mills sphalerons and non-abelian black holes discussed recently. New solutions possess quantized values of the dilaton charge.

Ten-dimensional heterotic string theory seems to provide a reasonable background for the discussion of the gravitational dynamics on a microscopic scale. Recently it was shown that an effective action for graviton, dilaton, axion and Yang-Mills fields resulting from vanishing of the relevant beta-functions possess interesting (4-dimensional) black hole solutions [1] that can shed new light onto the problem of a final stage of the Hawking evaporation [2] (for a review see e.g. [3]). New features of black holes in this model are related to the presence of a dilaton. As far as the Yang-Mills content is concerned, they belong to the $U(1)$ -sector, i.e. essentially *abelian*.

From the other hand, new *non-abelian* particle-like [4] and black hole [5] solutions were discovered in the pure Einstein-Yang-Mills (EYM) coupled system with the $SU(2)$ and higher rank groups [6]. They are generically unstable and satisfy (for the $SU(2)$ case) a non-abelian baldness theorem [7] stating that within the class of static asymptotically flat solutions only the embedded abelian ones can possess conserved (electric or magnetic) Coulomb charges. These features were further related to the *sphaleron* nature of the particle-like EYM solutions [8], [9]. The odd- n (number of zeros of the Yang-Mills potential) members of the family were shown to lie on the top of the potential barrier separating neighbouring topologically distinct Yang-Mills vacua [8]. General Morse-type argument in favor of this interpretation as well as some predictions concerning the number of unstable modes were given in [9].

A step forward to incorporate the EYM sphalerons into the realistic heterotic string context was made recently by Lavrelashvily and Maison [10] (see also [11]) who found numerically particle-like solutions of the Yang-Mills-dilaton coupled system in a flat space-time. The decoupling of gravity, however, corresponds to the unphysical limit of a characteristic parameter of the underlying string theory. Here we investigate the problem using the full bosonic part of the heterotic string effective action. We formulate (without proof) the corresponding non-abelian baldness conjecture and show (numerically) the existence of both regular and black hole static asymptotically flat solutions in the $SU(2)$ sector. They (necessarily) have dilaton charges and thus possess a Coulomb-type hair. This charge, however, is not a conserved one, but rather is a depended quantity which is determined by the distribution of the Yang-Mills field of the con-

figuration. Nevertheless, it is a new physical parameter describing the interaction between two sphalerons (black holes) due to the dilaton field. It is worth to be noted that Gibbons abelian dilatonic black holes [1] share the same property, that case a dilaton charge being expressible in terms of the electric and magnetic charges. We derive here a sum rule for the dilatonic charge which reduces to the Gibbons identity in the abelian case and which is also applicable to non-abelian configurations.

Other properties of the stringy non-abelian solutions are very similar to those of the EYM counterparts: they form a discrete sequence labeled by the number of nodes of the Yang-Mills potential and exist for an arbitrary value of the radius of the event horizon (regular solutions being a limiting case). They have vanishing Chern-Simons 3-forms both in the Lorentz and the Yang-Mills sectors and consequently a Kalb-Ramond field is not excited (although a purely topological axion charge is allowed).

We start with the following bosonic part of the 4-dimensional heterotic string effective action in the Einstein frame

$$S = \frac{1}{16\pi} \int \sqrt{-g} [m_{Pl}^2 (-R - 1/3 \exp(-4\Phi) H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 2\partial_\mu \Phi \partial^\mu \Phi) - \exp(-2\Phi) F_{a\mu\nu} F_a^{\mu\nu}] d^4x, \quad (1)$$

where Φ is the dilaton, $H_{\mu\nu\lambda}$ is the Kalb-Ramond field coupled to the Lorentz and the Yang-Mills Chern-Simons 3-forms $H = dB + \omega_{3L} - \omega_{3YM}$, $F_{a\mu\nu}$ is the Yang-Mills curvature corresponding to some gauge group containing $SU(2)$ as a subgroup (in what follows we shall deal with the $SU(2)$ component only). In terms of the Peccei-Quinn axion a

$$H^{\mu\nu\lambda} = 1/2 E^{\mu\nu\lambda\tau} \partial_\tau a \exp(4\Phi) \quad (2)$$

the action (1) reads

$$S = \frac{1}{16\pi} \int \sqrt{-g} [m_{Pl}^2 (-R + 2\partial_\mu \Phi \partial^\mu \Phi + 1/2 \partial_\mu a \partial^\mu a \exp(4\Phi)) - \exp(-2\Phi) F_{a\mu\nu} F_a^{\mu\nu} + a F_{a\mu\nu} \tilde{F}_a^{\mu\nu}] , \quad (3)$$

where \tilde{F} is the dual field strength. Using the identity $F_{a\mu\nu} \tilde{F}_a^{\mu\nu} = \nabla_\mu K^\mu$, where K^μ is the topological current dual to the Yang-Mills Chern-Simons 3-form, and eliminating the *total divergence*

one can cast the last term in the Eq.(2) into the form $\nabla_\mu a K^\mu$. After this an axion field will enter into the action only under the gradient and hence can be integrated out.

We are interested in the static spherically symmetric configurations which can conveniently be described by the line element

$$ds^2 = \frac{\Delta\sigma^2}{r^2}dt^2 - \frac{r^2}{\Delta}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (4)$$

The corresponding Yang-Mills connection after fixing the gauge can be parameterized in terms of two real-valued functions $W(r)$ (electric part) and $f(r)$ (magnetic part) as follows

$$gA_{a\mu}T^a dx^\mu = WL_r dt + (f-1)(L_\phi d\theta - L_\theta \sin\theta d\phi), \quad (5)$$

where $L_r = T^a n^a$, $L_\theta = \partial_\theta L_r$, $L_\phi = (\sin\theta)^{-1}\partial_\phi L_r$, $n^a = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$ is a unit vector, T^a are normalized hermitean generators of SU(2), and g is the coupling constant.

After dimensional reduction and elimination of a total divergence the action (3) will read

$$\begin{aligned} S = & \frac{1}{2} \int dt dr \{ m_P^2 l [\sigma'(\Delta/r - r) - (\Phi'^2 + a'^2 \exp(4\Phi))\Delta\sigma] + \\ & + g^{-2} [(\frac{W'^2 r^2}{\sigma} + \frac{2f^2 r^2 W^2}{\Delta\sigma} - \frac{2f'^2 \Delta\sigma}{r^2} - \frac{\sigma(1-f^2)^2}{r^2}) \exp(-2\Phi) - 2a'W(f^2 - 1)] \}, \end{aligned} \quad (6)$$

where primes denote the derivatives with respect to r . In the abelian sector $f \equiv 0$ it is equivalent to that considered in [1] and generates the embedded abelian solutions similar to Gibbons electrically and magnetically charged black holes with long-range dilaton hair. Within the abelian sector there exist generalized duality transformations which allow to eliminate an axion field a , or, inversely, to generate solutions with axion hair from Maxwell-dilaton ones.

The Yang-Mills part of the action (6) is close to that of the pure EYM system [4], [5], [7] and can be analysed along the same lines. We formulate here without proof the main result as the *generalized non-abelian baldness* theorem: *among all asymptotically flat configurations (both regular or black hole) only embedded abelian ones can have electric or magnetic charges.* Moreover, one can show that solutions with $W \neq 0$ necessarily have an electric charge. So, to investigate the essentially non-abelian solutions, we are led to consider only $W \equiv 0$ case. Then

it can be easily shown that the axion hair have to vanish apart from the purely topological hair of the type [12]. Indeed, a variation of (6) with respect to a leads to the following equation

$$a' = \frac{2R_g^2 W(1-f^2) \exp(-4\Phi)}{\Delta\sigma} + a_1, \quad (7)$$

where $R_g = 1/(m_{Pl}g)$ is the characteristic length, and $a_1 = \text{const}$ results after the first integration. For $W = 0$ the solution reads

$$a = a_0 + a_1 r, \quad (8)$$

where $a_0 = \text{const}$. From the asymptotic flatness $a_1 = 0$, while a_0 can still be non-zero and be interpreted as a topological axion charge [12]. Obviously such an axion hair is fully non-dynamical and has no influence on the remaining fields configuration.

In the pure magnetic sector the variation of the action (6) gives the following set of equations for f, Φ, σ and Δ :

$$\left(\frac{f' \Delta \sigma \exp(-2\Phi)}{r^2}\right)' = \frac{\sigma f(f^2 - 1) \exp(-2\Phi)}{r^2}, \quad (9)$$

$$(\Phi' \Delta \sigma)' = -\frac{R_g^2 \sigma F \exp(-2\Phi)}{r^2}, \quad (10)$$

$$(\ln \sigma)' = \frac{2R_g^2 f'^2 \exp(-2\Phi)}{r} + r \Phi'^2, \quad (11)$$

$$-\left(\frac{\Delta}{r}\right)' + 1 = \frac{R_g^2 F \exp(-2\Phi)}{r^2} + \Delta \Phi'^2, \quad (12)$$

where

$$F = 2\Delta f'^2 + (1 - f^2)^2. \quad (13)$$

The Yang-Mills equation (9) is scale-invariant and remains the same under the constant shift of the dilaton $\Phi \rightarrow \Phi + c$. All other equations recover their initial form after the accompanying rescaling $R_g \rightarrow R_g e^c$. It is convenient to fix the scale by imposing on the dilaton field an asymptotic condition $\Phi(\infty) = 0$. Then it can be shown that the leading term for the dilaton at infinity will be a Coulomb one

$$\Phi = \frac{D}{r} + O\left(\frac{1}{r}\right) \quad (14)$$

provided the Yang-Mills field is not in the vacuum state $|f| \equiv 1$. It can be easily seen from the Eq.(11) that the dilaton charge D dominates the asymptotic behavior of the metric function σ

$$\sigma = 1 - \frac{D^2}{2r^2} + O\left(\frac{1}{r^4}\right). \quad (15)$$

The asymptotic of the second metric function Δ has the form

$$\Delta = r^2 - 2Mr + D^2 + O\left(\frac{1}{r}\right) \quad (16)$$

where M is the Schwarzschild mass. It should be noted that the contributions of the dilaton charge squared is canceled in the asymptotic behavior of the metric component

$$g_{00} = \frac{\Delta\sigma^2}{r^2} = 1 - \frac{2M}{r} + O\left(\frac{1}{r^3}\right) \quad (17)$$

and hence the dilaton charge does not produce the Reissner-Nordstrom type contribution to the asymptotic form of a metric.

An asymptotic behavior of the Yang-Mills function f compatible with the asymptotic flatness is either $f(\infty) = 0$ which corresponds to the magnetically charged configuration (abelian), or $f = \pm 1$ (non-charged sphaleronic configuration) [7], [8]. Here we concentrate on the second case.

To specify the behavior of variables on the other side of the r -semiaxis it is convenient to introduce the parameter r_0 which is zero for the regular sphaleron solution and is equal to the horizon radius r_H in the black hole case. In both cases $\Delta(r_0) = 0$. The other functions have at this point finite and non-zero values. Then, integrating the dilaton equation (10) from r_0 to infinity we obtain the following sum rule for the dilaton charge

$$D = R_g^2 \int_{r_0}^{\infty} \frac{\sigma F \exp(-2\Phi)}{r^2} dr \quad (18)$$

Since the function F given by the Eq.(13) is positive definite (as well as σ), the dilaton charge is non-zero for both the abelian $f \equiv 0$, $\sigma = (1 + D^2/r^2)^{-1/2}$ and the non-abelian configurations. In the first case the Eq.(18) reproduces the result of Gibbons [1] $D \sim (\text{magnetic charge})^2$.

An analogous sum rule can be obtained for the Schwarzschild mass by integrating the Eq.(12)

$$M = \frac{1}{2} \int_{r_0}^{\infty} \left(\Delta \Phi'^2 + \frac{F \exp(-2\Phi)}{r^2} \right) dr + M_0 \quad (19)$$

where $M_0 = 0$ in the regular case and $M_0 = M_H$ - "bare" black hole mass in the black hole case.

An equation for the metric function σ (11) can be integrated as follows

$$\sigma = \exp\left[-\int_r^\infty \left(\frac{2R_g^2 f'^2 \exp(-2\Phi)}{r} + r\Phi'^2\right)dr\right] \quad (20)$$

and then substituted into the other equations (9), (10), (12). Transforming to a new variable $\rho = r^2$ one obtains finally the following set of coupled equations for three quantities f , Φ and Δ

$$\Delta(f_{\rho\rho} - 2f_\rho\Phi_\rho) + \frac{1}{2}Gf_\rho + \frac{f(1-f^2)}{4\rho} = 0, \quad (21)$$

$$\Delta(\Phi_{\rho\rho} + \frac{1}{\rho}\Phi_\rho) + \frac{1}{2}G\Phi_\rho + \frac{R_g^2 F \exp(-2\Phi)}{4\rho^2} = 0, \quad (22)$$

$$\Delta_\rho + \Delta(2\rho\Phi_\rho^2 - \frac{1}{2\rho}) + \frac{1}{2}\left(\frac{R_g^2 F \exp(-2\Phi)}{\rho} - 1\right) = 0, \quad (23)$$

where

$$G = 1 - \frac{R_g^2(1-f^2)^2 \exp(-2\Phi)}{\rho}. \quad (24)$$

First we turn to the discussion of the regular solutions. A series expansion of the solution of the system (21)-(23) in the vicinity of the origin can be written as follows

$$f = -1 + bx + O(x^2), \quad (25)$$

$$\Phi = \Phi_0 - 2b^2 x \exp(-2\Phi_0) + O(x^2), \quad (26)$$

$$\frac{\Delta}{\rho} = 1 + O(x) \quad (27)$$

where $x = \rho/R_g^2$ and Φ_0 is the (generally non-zero) value of the dilaton field at the origin. Recall that we have fixed an overall scale by imposing the condition $\Phi(\infty) = 0$, after what the length parameter R_g in the Eqs.(21)-(23) have been absorbed by passing to a dimensional variable x . The system consists of two equations of the second order and one of the first order, hence the solution will be fixed completely by the boundary conditions for $f(0)$, $f'(0)$, $\Phi(0)$, $\Phi'(0)$ and $\Delta(0)$, which are parameterized according to Eqs. (25)-(27) in terms of b and Φ_0 . Alternatively, the problem can be thought of as the Sturm-Liouville problem with fixed $f(0) = -1$, $\Delta/\rho \rightarrow 1$ as $\rho \rightarrow 0$, $|f(\infty)| = 1$, $\Phi(\infty) = 0$ and $\Delta/\rho \rightarrow 1$ as $\rho \rightarrow \infty$. The solution, like Bartnik-McKinnon solution [4] for the pure EYM system, exists for discrete values of the parameters b and Φ_0 ,

labeled by the number of zeros n of the Yang-Mills function f . For each n there exist one pair of values of b and Φ_0 , hence the family of solutions remains one-parametric as in the EYM case [4]. These values found numerically for some lower n are shown in the tab.1 together with the corresponding values of the total mass and the dilaton charge as given by Eqs. (18), (19). With increasing n both these quantities are likely to tend to some limiting values. Numerical solutions for f , Φ , and σ are shown at the figs.1-3, behavior of the mass function is depicted at the fig.4. Note that functions $\Phi(r)$ and $\sigma(r)$ are monotonic as can be anticipated from the Eqs.(20)-(23). It is worth to note that like in the abelian case [1] a dilaton charge is not an independent parameter. Note that numerically M and D (in the units $R_g = 1$) are very close together.

Table 1

n	b	Φ_0	σ_0	M	D
1	1.0718	0.9300	0.3936	0.5777	0.5782
2	8.3612	1.7923	0.1665	0.6850	0.6852
3	53.8351	2.6320	0.0678	0.7035	0.7042

Physical interpretation of the regular solutions can be given along the lines of [8]. A path in the functional space connecting the Yang-Mills vacua with neighbouring winding numbers can be constructed using the parameterization of the Yang-Mills connection similar to given in [8] adding to it an appropriate dilaton path. Odd- n solutions then can be shown to play the role of sphalerons. For all n the solutions are expected to be unstable with one of the instability modes being the rolling down mode from the top of the potential barrier which separates the Yang-Mills vacua.

In the black hole case an additional continuous real parameter $\rho_H = r_H^2$, with r_H being the largest (simple) zero of the function Δ , introduces an independent length scale. The whole family of solutions will then be two-parametric in terms of (r_H, n) , $x = \rho/\rho_H$ will be a new

radial variable. A series expansion of the solution in the outside vicinity of the horizon reads

$$f = f_H + \frac{f_H(f_H^2 - 1)}{2G_H} y + O(y^2), \quad (28)$$

$$\Phi = \Phi_H + \frac{y}{2} \left(1 - \frac{1}{G_H}\right) + O(y^2), \quad (29)$$

$$\Delta = \frac{y}{2} \rho_H G_H (1 + O(y)), \quad (30)$$

where $G_H = G(\rho_H)$ and $y = x - 1$. The solutions have different character depending on the value of the parameter r_H . Likely to the pure EYM case [5], in the limit $r_H \rightarrow 0$ the solution outside the horizon is indistinguishable from the regular one. In the opposite limit $r_H \rightarrow \infty$ the Yang-Mills equation decouples from the coupled gravity-dilaton system, and likely to the pure EYM case one could anticipate the existence of the (fully non-linear) Yang-Mills equation on the black hole background. It can be easily seen from the expansion (29) that the derivative of the dilaton field at the horizon tends to zero and hence in this limit the dilaton is not excited at all (otherwise regular black hole solution with dilaton but without Yang-Mills hair could not exist). So in the limit $r_H \rightarrow \infty$ the solution reduces to that of the EYM system [6] (in particular, an analytic form of the solution for $n = 1$ is known).

For $r_H \sim 1$ the solution was studied numerically. With fixed x_H it exists for the discrete values of the parameters f_H and Φ_H . These values for $r_H = 1$ are shown in the tab. 2 (with the same meaning of n as before) together with the horizon values of σ , the field mass $M - M_H$ and the dilaton charge. Figures 5–7 depict the corresponding numerical curves. The similarity with the EYM black holes [5] is manifest. Both the field mass and the dilaton charge are rapidly saturated with increasing n .

Table 2

n	f_H	Φ_H	σ_H	M	$M - M_H$	D	T/T_H
1	-0.5937	0.4420	0.7850	0.8368	0.3368	0.5124	0.6490
2	-0.1321	0.5445	0.8598	0.8651	0.3651	0.5749	0.5804
3	-0.0218	0.5493	0.8653	0.8658	0.3658	0.5765	0.5771

To illustrate the difference between the metrics of the non-abelian and the abelian magnetically charged solution [1] one can define the metric "magnetic charge" function $P^2(r)$ as follows

$$g_{00} = 1 - 2M/r\sqrt{1 + D^2/r^2} + P^2/r^2, \quad (31)$$

such that $P = \text{const}$ (magnetic charge) in the abelian case. The corresponding curves for non-abelian stringy black holes are shown at the fig. 8.

We have also calculated the Hawking temperature

$$\begin{aligned} T &= \frac{\sigma(r_H) \left(\frac{d\Delta}{dr}\right)_{r=r_H}}{4\pi r_H^2} \\ &= \frac{\sigma(r_H) G_H}{4\pi r_H}. \end{aligned} \quad (32)$$

The ratio of the Hawking temperature to the temperature of the Schwarzschild black hole possessing the same radius of the event horizon $T_H = 1/4\pi r_H$ is shown in the tab. 2. We will discuss the thermodynamics of non-abelian stringy black holes in a separate publication.

To summarize: we have found strong numerical evidence in favour of the existence of both regular and black hole 4-dimensional solutions of the heterotic string effective action in the $SU(2)$ sector. They generalize the corresponding EYM sphalerons and non-abelian black holes, are unstable, and are supposed to play a similar role in the context of the quantum theory. They have quantized values of the dilaton charge and interact among themselves by both gravitational and dilaton long-range forces. Spherically symmetric solutions have zero both Lorentz and Yang-Mills Chern-Simons 3-forms and, correspondingly, a Kalb-Ramond field is not excited (though topological axion hair is not excluded).

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Figure Captions

Fig. 1. Yang-Mills magnetic field function f , regular case.

Fig. 2. Dilaton field, regular case.

Fig. 3. Metric function σ , regular case.

Fig. 4. Mass distribution for $n = 2$ regular solution: A – total mass, B – Yang-Mills contribution (second term in the Eq. (19)), C – dilaton contribution (first term in the Eq. (19)).

Fig. 5. Yang-Mills function f for the $r_H = 1$ black holes.

Fig. 6. Dilaton field for $r_H = 1$ black hole.

Fig. 7. Metric function σ for $r_H = 1$ black hole.

Fig. 8. Metric "charge" function $P^2(r)$.